

DETERMINISTIC VERSUS STOCHASTIC ASPECTS OF SUPEREXPONENTIAL POPULATION GROWTH MODELS

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ABSTRACT. Deterministic population growth models with power-law rates can exhibit a large variety of growth behaviors, ranging from algebraic, exponential to hyperexponential (finite time explosion). In this setup, selfsimilarity considerations play a key role, together with two time substitutions. Two stochastic versions of such models are investigated, showing a much richer variety of behaviors. One is the Lamperti construction of selfsimilar positive stochastic processes based on the exponentiation of spectrally positive processes, followed by an appropriate time change. The other one is based on stable continuous-state branching processes, given by another Lamperti time substitution applied to stable spectrally positive processes.

Keywords: population growth models, selfsimilarity, Lamperti transforms and processes.

1. INTRODUCTION

Deterministic population growth models (1) with power-law rates μx^γ , $\mu > 0$, can exhibit a large variety of behaviors, ranging from algebraic ($\gamma < 1$), exponential ($\gamma = 1$) to hyperexponential (finite time explosion if $\gamma > 1$) growth for the size (or mass) $x(t)$ of some population at time $t \geq 0$. The exponential (Malthusian) growth regime with $\gamma = 1$ discriminates between the two other ones and the transition at $\gamma = 1$ is quite sharp. In this setup, selfsimilarity considerations (with Hurst index $\alpha = 1/(1 - \gamma)$) play a key role, together with two time substitutions. Log-selfsimilarity considerations can also be introduced while exponentiating the latter model for $x(t)$. In this setup, the discriminating process grows at double (or superexponential) speed. This discriminating process separates two log-self-similar processes, one growing at exp-algebraic rate and the other one blowing-up in finite time.

In this manuscript, two stochastic versions of such population growth models with similar flavor are investigated, showing a much richer variety of behaviors. One is the Lamperti construction of selfsimilar positive stochastic processes based on the exponentiation of spectrally positive processes, followed by an appropriate time change. As an example, the Lamperti diffusion process (22) is studied in some detail, including the noncritical cases. For the critical case with $\mu = 0$ for instance, we show that the transition $\gamma < 1$ to $\gamma > 1$ is rather smooth: indeed, if $\gamma < 1$, state ∞ is a natural inaccessible boundary whereas state 0 is exit (or absorbing) and reached eventually in finite time. The population dies out (extinction) fast. If $\gamma = 1$ (when the discriminating critical process is geometric Brownian motion), state ∞ is an entrance state and state 0 a natural inaccessible boundary. State 0

(extinction) is reached eventually but now not in finite time. If $\gamma > 1$, state 0 is a natural inaccessible boundary whereas state ∞ is an entrance state. The process drifts to ∞ (explosion) but not in finite time. Situations for which there is a finite time explosion can occur but only in noncritical cases when $\gamma > 1$ and μ exceeds some positive threshold. In all cases, depending on $\gamma < 1$ ($\gamma > 1$), such processes are stochastically selfsimilar with Hurst index $\alpha > 0$ ($\alpha < 0$).

The other one is based on continuous-state branching processes (CSBPs) $x(t)$, as given by another Lamperti time substitution of spectrally positive processes: in this respect, the a -stable Lamperti CSBP (with $a \in (1, 2)$) and the one-sided a -stable CSBP (with $a \in (0, 1)$) are investigated in some detail. Both noncritical and critical cases are considered. The critical version of these models are shown to exhibit self-similarity properties: the obtained Hurst indices are $\alpha = 1/(a - 1)$ with range $\alpha > 1$ and $\alpha < -1$, respectively. Taking $a \rightarrow 1^\pm$ yields in the first place the deterministic Malthusian growth model: $x(t) = xe^{(\mu \pm \kappa)t}$. This Malthusian regime separates a situation for which $\mathbf{E}(x(t) | x(t) > 0) \propto t^\alpha$ has superlinear algebraic growth rate (for the a -Lamperti model) and a situation for which $x(t)$ is not regular as it blows up for all time $t > 0$ (for the one-sided a -stable model). The Malthus model is the discriminating critical process of such CSBP population growth models and the situation looks quite similar to the deterministic setup, although much more complex. The transition at $a = 1$ is sharp. While considering a different limiting process as $a \rightarrow 1^\pm$, we obtain the Neveu CSBP model which grows a.s. at double superexponential speed. The critical version of this process is no longer self-similar. It plays the role of the superexponential discriminating deterministic model separating two log-self-similar models: the exp-algebraic and the blowing-up regimes, respectively.

2. DETERMINISTIC POPULATION GROWTH MODELS

2.1. A class of self-similar growth models. Let $x(t) \geq 0$ denote the size (mass) of some population at time $t \geq 0$, with initially $x := x(0) \geq 0$. With $\mu, \gamma > 0$, consider the growth dynamics

$$(1) \quad \dot{x}(t) = \mu x(t)^\gamma, \quad x(0) = x,$$

for some velocity field $v(x) = \mu x^\gamma$. Integrating when $\gamma \neq 1$ (the non linear case), we get formally

$$(2) \quad x(t) = \left(x^{1-\gamma} + \mu(1-\gamma)t \right)^{1/(1-\gamma)}.$$

Three cases arise:

- $0 < \gamma < 1$: then $x \geq 0$ makes sense and in view of $1/(1-\gamma) > 1$, the growth of $x(t)$ is algebraic at rate larger than 1. We note that $x(t, x) := x(t)$ with $x(0) = x$ obeys the selfsimilarity property: for all $\lambda > 0$, $t \geq 0$ and $x \geq 0$, $x(\lambda t, \lambda^\alpha x) = \lambda^\alpha x(t, x)$, with $\alpha := 1/(1-\gamma) > 1$, the Hurst exponent. When $x = 0$, the dynamics has two solutions, one $x(t, 0) \equiv 0$ for $t \geq 0$ and the other $x(t, 0) = (\mu(1-\gamma)t)^{1/(1-\gamma)}$ because the velocity field v in (1) with $v(0) = 0$, is not Lipschitz as x gets close to 0, having an infinite derivative. The solution $x(t, 0) = (\mu(1-\gamma)t)^{1/(1-\gamma)}$ with $x = 0$ reflects some spontaneous generation phenomenon: following this path, the mass at time $t > 0$ is not 0, although initially it was.

• $\gamma > 1$: then $x > 0$ only makes sense and explosion or blow-up of $x(t)$ occurs in finite time $t_{\text{exp}} = x^{1-\gamma} / [\mu(\gamma - 1)]$. Up to the explosion time t_{exp} , $x(t)$ is selfsimilar with Hurst exponent $\alpha = 1/(1 - \gamma) < 0$. Whenever $x(t)$ blows up in finite time, following [31], we shall speak of an hyperexponential growth regime. This model was shown meaningful as a world population growth model over the last two millenaries, [31]. There is also some recent empirical interest into models with similar behavior in [29], [13] and [14]. The finite-time explosion feature, the related interpretation problems and the previous works about this interpretation have been emphasized in [25], where the author considers the technological advance of a given market. More technically, necessary and sufficient conditions for the existence of such a blowing up regime involving the asymptotic form of the local series representation for the general solutions around the singularities are given in [8].

• $\gamma = 1$: this is a simple special case not treated in (2), strictly speaking. However, expanding the solution (2) in the leading powers of $1 - \gamma$ yields consistently:

$$(3) \quad \begin{aligned} x(t) &= e^{\log(x^{1-\gamma} + \mu(1-\gamma)t)/(1-\gamma)} \\ &= e^{\log[x^{1-\gamma}(1 + \mu x^{\gamma-1}(1-\gamma)t)]/(1-\gamma)} \sim x e^{(1/(1-\gamma))\mu x^{\gamma-1}(1-\gamma)t} \sim x e^{\mu t}. \end{aligned}$$

Here $x \geq 0$ makes sense for (1) with $x(t) = x e^{\mu t}$ for $t \geq 0$ if $x \geq 0$. This is the simple Malthus growth model. The Malthus regime with $\gamma = 1$ will be called “discriminating” for (1), in the sense that it separates a slow algebraic growth regime ($\gamma < 1$) and a blowing-up regime ($\gamma > 1$).

Remark: (i) One can extend the range of γ as follows: if $\gamma = 0$, for all $x \geq 0$, $x(t) = x + \mu t$, a linear growth regime. If $\gamma < 0$, (2) holds for all $x \geq 0$: because $1/(1 - \gamma) < 1$ the growth of $x(t)$ is again algebraic but now at rate smaller than 1. When $\gamma \leq 0$, the spontaneous generation phenomenon also holds with the velocity field itself diverging near $x = 0$ if $\gamma < 0$: the solution $x(t) \equiv 0$ for $t \geq 0$ is no longer valid. For this range of γ , $x(t) := x(t, x)$ obeys the selfsimilarity property with Hurst exponent $\alpha = 1/(1 - \gamma) \in (0, 1]$.

(ii) One can also extend the range of μ as follows: if $\mu < 0$, depending on $0 < \gamma < 1$ or $\gamma > 1$, the process either goes extinct in finite time $t_{\text{ext}} = x^{1-\gamma} / [\mu(\gamma - 1)]$ or decays at algebraic rate $1/(1 - \gamma)$ reaching 0 in infinite time (respectively). Because growth is our main interest, we shall avoid this case in general.

2.2. Time-changes. We shall consider two different kinds of time substitution which shall prove of interest to us.

(i) Consider the trivial dynamics $\dot{s}(\tau) = \mu$, with $s(\tau) = s(0) + \mu\tau$, for some clock-time $\tau \geq 0$. Let $y(\tau) = \exp s(\tau)$. Then $\dot{y}(\tau) = \mu y(\tau)$, $y(0) = \exp s(0) > 0$, with $y(\tau) = y(0) e^{\mu\tau} > 0$. Consider the time substitution: $t_\tau = \int_0^\tau y(\tau')^{1/\alpha} d\tau'$. Then its inverse is $\tau_t = \int_0^t x(s)^{-1/\alpha} ds$ where $x(t) = y(\tau_t)$. The dynamics of $x(t)$ is

$$(4) \quad \dot{x}(t) = \dot{y}(\tau_t) \dot{\tau}_t = \mu x(t)^{1-1/\alpha}.$$

It coincides with (1) provided $\alpha = 1/(1 - \gamma)$ or $\gamma = 1 - 1/\alpha$. Thus $x(t)$ in (1) is a time-changed version of $y(\tau) = \exp s(\tau)$.

(ii) If $s(\tau) > 0$ for all $\tau \geq 0$ (requiring $s := s(0) > 0$ and $\mu > 0$), the process $y(t)$ is itself a time-changed version of $s(\tau)$. Consider indeed the time substitution: $t_\tau = \int_0^\tau s(\tau')^{-1} d\tau'$. Then its inverse is $\tau_t = \int_0^t y(s) ds$ where $y(t) = s(\tau_t)$. The dynamics of $y(t)$ is

$$(5) \quad \dot{y}(t) = \dot{s}(\tau_t) \dot{\tau}_t = \mu y(t).$$

If $s > 0$, $\mu < 0$, this is true only up to the time when $s(\tau)$ first hits zero.

2.3. Exponentiating and log-selfsimilarity. Finally, with $\mu, \gamma > 0$, consider now the dynamics

$$(6) \quad \dot{z}(t) = \mu z(t) (\log z(t))^\gamma, \quad z(0) = z.$$

Introducing $x(t) = \log z(t)$ and $x = \log z$, $x(t)$ obeys (1). Integrating (6), we get formally if $\gamma \neq 1$

$$(7) \quad z(t) = \exp \left((\log z)^{1-\gamma} + \mu(1-\gamma)t \right)^{1/(1-\gamma)}.$$

We conclude:

- $0 < \gamma < 1$: the integrated solution makes sense only when $z \geq 1$ in which case the growth of $z(t)$ is exp-algebraic at algebraic rate $1/(1-\gamma) > 1$. We note that with $z(t, z) := z(t)$ and $z(0) = z$, $\log z(t, z) := x(t, x)$ obeys the self-similarity property with Hurst exponent $\alpha = 1/(1-\gamma) > 1$. So $z(t)$ is log-selfsimilar.
- $\gamma > 1$: then $z > 1$ only makes sense in general and explosion or blow-up of $z(t)$ occurs in finite time $t_{\text{exp}} = (\log z)^{1-\gamma} / [\mu(\gamma-1)]$. Up to the explosion time t_{exp} , $z(t)$ is log-selfsimilar with Hurst exponent $\alpha = 1/(1-\gamma) < 0$. If $\gamma > 1$ is an integer, values of $z < 1$ are admissible.
- $\gamma = 1$: then $z \geq 0$ makes sense for (6) with superexponential solution $z(t) = ze^{\mu t}$ for $t \geq 0$. If $z < 1$, $z(t)$ decays at double exponential (or superexponential) pace, whereas if $z > 1$ growth occurs at superexponential (or double exponential) pace, with $z(t) \equiv 1$ if $z = 1$. $\gamma = 1$ is discriminating for (6) again separating a growth regime at exp-algebraic rate and a blowing-up regime.

One can extend the range of γ as follows: if $\gamma = 0$, $z(t) = ze^{\mu t}$, the Malthusian exponential growth regime. If $\gamma < 0$, (7) holds for all $z > 0$: because $1/(1-\gamma) < 1$, the growth of $z(t)$ is exp-algebraic with time now at algebraic rate smaller than 1 and $z(t)$ is log-selfsimilar with Hurst exponent $\alpha = 1/(1-\gamma) \in (0, 1]$.

3. STOCHASTIC VERSION OF THE SELF-SIMILAR GROWTH PROCESS

We now investigate a natural Markovian stochastic version of the positive self-similar growth process which was first designed in [18]. They are obtained while considering in the latter construction a much richer class of driving processes $s(\tau)$: the class of spectrally positive processes with stationary independent increments. A different attempt to the stochastization of the finite-time singularity effect was designed in [30] and applied to the space-time clustering events and power law Gutenberg-Richter distribution of earthquake energies.

3.1. Spectrally positive process with stationary independent increments.

We start with the construction of a spectrally positive process with stationary independent increments, [16].

Let c and $b > 0$ be two constants. Let $s(\tau)$ with $s := s(0)$ be a spectrally positive process with independent increments and infinitesimal generator acting on $\phi \in C^2$, [4],

$$(8) \quad G\phi(s) = \lim_{\tau \rightarrow 0+} \frac{\mathbf{E}_s \phi(s(\tau)) - \phi(s)}{\tau} = \int_0^\infty (\phi(s+v) - \phi(s) - v\phi'(s)1_{\{v \leq 1\}}) \pi(dv) + c\phi'(s) + \frac{1}{2}b^2\phi''(s).$$

π is the Lévy measure of the jumps of $s(\tau)$, whose support is restricted to the positive half-line; π is assumed to integrate $1 \wedge v^2$. We also assume $\pi(dv) = \rho(v)dv$ for some density function $\rho(v)$. $s(\tau)$ started at s has a drift term $c\tau$ and a Brownian component w with constant local standard deviation $b > 0$ and a pure random jump measure term N with intensity $ds \cdot \pi(dv)$, specifically:

$$(9) \quad s(\tau) = s + c\tau + bw(\tau) + \int_0^\tau \int_0^\infty vN(ds, dv).$$

If $s > 0$ and $c < 0$, whenever $s(\tau)$ becomes negative, it will do so while hitting the origin. Taking $\phi(s) = e^{-ps}$, $p \geq 0$, $\mathbf{E}_s \phi(s(\tau)) = \mathbf{E}_s e^{-ps(\tau)}$ is the Laplace-Stieltjes transform (LSt) of $s(\tau)$ and

$$(10) \quad G\phi(s) = -e^{-ps}\psi(p)$$

where

$$(11) \quad \psi(p) = \int_0^\infty (1 - e^{-pv} - pv1_{\{v \leq 1\}}) \pi(dv) + cp - \frac{1}{2}b^2p^2$$

is the log-Laplace exponent of $s(\tau)$. As required therefore for Markov processes with stationary independent increments,

$$(12) \quad \mathbf{E}_s e^{-p(s(\tau)-s)} = e^{-\tau\psi(p)}.$$

The function $\psi(p)$ is concave with $\psi'(0) = \int_1^\infty v\pi(dv) + c =: \mu$.

3.2. Exponential of the spectrally positive process.

We now turn to taking the exponential of the spectrally positive process $s(\tau)$.

Let $y(\tau) = \exp s(\tau)$, exponentiating $s(\tau)$. Then $y(\tau)$ with $y := y(0)$ is a multiplicative Markov process with infinitesimal generator ($\psi(y) = \phi(\log y)$),

$$(13) \quad \tilde{G}\psi(y) := \lim_{\tau \rightarrow 0+} \frac{\mathbf{E}_y \psi(y(\tau)) - \psi(y)}{\tau} = \int_1^\infty (\psi(yu) - \psi(y) - y \log u \psi'(y) 1_{\{u \leq e\}}) \tilde{\pi}(du) + (\frac{1}{2}b^2 + c) y \psi'(y) + \frac{1}{2}b^2 y^2 \psi''(y).$$

$\tilde{\pi}$ is the Lévy measure of the jumps of $y(\tau)$ started at y , supported by $(1, \infty)$, with $\tilde{\pi}(du) = u^{-1} \rho(\log u) du$, the image measure of π under the exponential transformation. If $\psi(y) = y^q$,

$$(14) \quad \tilde{G}\psi(y) := y^q \left(\int_1^\infty (u^q - 1 - q \log u 1_{\{u \leq e\}}) u^{-1} \rho(\log u) du + cq + \frac{1}{2}b^2 q^2 \right) =: y^q \xi(q),$$

leading to

$$(15) \quad \mathbf{E}_y \left(\frac{y(\tau)}{y} \right)^q = e^{\tau \xi(q)}$$

for all $q : \xi(q)$ exists. The idea of a multiplicative process is already present in [1], [21]. In these papers, a non-linear version of the multiplicative model was used (with an extra positive feedback not introduced here) to model explosive financial bubble prices.

Examples:

- If $\rho \equiv 0$ (no jumps for $s(\tau)$), \tilde{G} is the infinitesimal generator of the Itô diffusion process ($\mu = b^2/2 + c$):

$$(16) \quad dy(\tau) = \mu y(\tau) d\tau + by(\tau) dw(\tau) = y(\tau) (\mu d\tau + b dw(\tau)),$$

with $w(\tau)$ the standard Brownian motion. We have $y(\tau) = e^{s(\tau)}$ with $s(\tau)$ obeying: $ds(\tau) = cd\tau + b dw(\tau)$. In the exponentiation process, we are led to a Malthus equation for y with randomized rate $\mu d\tau \rightarrow \mu d\tau + b dw(\tau)$.

- Let $b = 0$ and $\rho(v) = \kappa v^{-(1+a)}/\Gamma(-a)$, $\kappa > 0$ and $a \in (1, 2)$. Then

$$(17) \quad \tilde{G}\psi(y) := \frac{\kappa}{\Gamma(-a)} \int_1^\infty (\psi(yu) - \psi(y) - y \log u \psi'(y) 1_{\{u \leq e\}}) u^{-1} (\log u)^{-(1+a)} du + cy\psi'(y).$$

If $\psi(y) = y^q$,

$$(18) \quad \tilde{G}\psi(y) := y^q \left(\frac{\kappa}{\Gamma(-a)} \int_1^\infty (u^q - 1 - q \log u 1_{\{u \leq e\}}) u^{-1} (\log u)^{-(1+a)} du + cq \right),$$

leading to

$$(19) \quad \mathbf{E}_y \left(\frac{y(\tau)}{y} \right)^q = e^{\tau \xi(q)}$$

where

$$(20) \quad \xi(q) = \frac{\kappa}{\Gamma(-a)} \int_1^\infty (u^q - 1 - q \log u 1_{\{u \leq e\}}) u^{-1} (\log u)^{-(1+a)} du + cq.$$

3.3. Lamperti time substitution. The self-similar process of interest is now a time-changed version of $y(\tau)$.

Following the path of (i) in Subsection 2.2, let $x(t) = y(\tau_t)$ be a time-changed version of $y(\tau)$ using the (now random) time substitution: $t_\tau = \int_0^\tau y(\tau')^{1/\alpha} d\tau'$ and its inverse $\tau_t = \int_0^t x(s)^{-1/\alpha} ds$. Then $x(t)$ with $x := x(0)$ is a Markov process with infinitesimal generator

$$(21) \quad L\psi(x) := \lim_{t \rightarrow 0^+} \frac{\mathbf{E}_x \psi(x(t)) - \psi(x)}{t} = x^{-1/\alpha} \tilde{G}\psi(x) = \int_1^\infty (\psi(xu) - \psi(x) - x \log u \psi'(x) 1_{\{u \leq e\}}) \tilde{\pi}_x(du) + \left(\frac{1}{2}b^2 + c \right) x^{1-1/\alpha} \psi'(x) + \frac{1}{2}b^2 x^{2-1/\alpha} \psi''(x).$$

$\tilde{\pi}_x$ is the Lévy measure of the jumps of $x(t)$ with support $(1, \infty)$ and with $\tilde{\pi}_x(du) = x^{-1/\alpha} u^{-1} \rho(\log u) du$, given $x(t)$ is in state x . Putting $\gamma = 1 - 1/\alpha$ and $\mu = b^2/2 + c$, the drift term is μx^γ as in (1). Note that, depending on $c < -b^2/2$ ($\mu < 0$) or $c > -b^2/2$ ($\mu > 0$), the drift term of $x(t)$ is either negative or positive.

It holds under some general conditions that for all $\lambda > 0$, $t \geq 0$ and $x \geq 0$, $\{x(\lambda t, \lambda^\alpha x)\} \stackrel{d}{=} \lambda^\alpha \{x(t, x)\}$, with $\alpha := 1/(1 - \gamma)$, [18]. The stochastic process $x(t)$ is selfsimilar with Hurst index α , using a terminology employed in [15] and [22].

3.4. Examples. We shall first study the purely diffusive case in some details.

(i) If $\rho \equiv 0$ (no jumps for $x(t)$) a stochastic extension of (1) with continuous sample paths is ($\mu = b^2/2 + c$)

$$(22) \quad \begin{aligned} dx(t) &:= \mu x(t)^\gamma dt + bx(t)^{(1+\gamma)/2} dw(t) \\ &= x(t)^\gamma \left(\mu dt + bx(t)^{(1-\gamma)/2} dw(t) \right), \quad x(0) = x > 0. \end{aligned}$$

Here $w(t)$ is the standard Brownian motion. The latter Lamperti stochastic differential equation is a time-changed version of the Itô diffusion (16). Lamperti [18] only considered (22) with $\gamma < 1$. Such models were considered in [29]. They are in the class of the so-called generalized CEV diffusion processes (see [20] to detect and analyze financial bubbles and [24]), whose drift and local volatility terms $f(x) := \mu x^\gamma$ and $g(x) := bx^{(1+\gamma)/2}$ obey $f = Kgg'$ for some constant $K = 2\mu/(b^2(1+\gamma))$, a possible signature of power-law stationary distribution, [24]. Some authors (see [3] for instance) considered a similar SDE but in the sense of Stratonovitch. Although interesting, such SDEs fail to be self-similar.

The invariant or speed measure density is $m(x) = g^{-2}(x) \exp 2 \int^x f/g^2(y) dy = b^{-2} x^{2\mu/b^2 - (1+\gamma)}$. The (non-decreasing) scale or harmonic function is

$$(23) \quad \phi(x) = A + B \int^x dy \exp \left(-2 \int^y f(z)/g^2(z) dz \right),$$

for some constants $A, B > 0$, so with $\phi'(x) = B \exp(-2 \int^x f(y)/g^2(y) dy) = Bx^{-2\mu/b^2} > 0$. It is such that $\phi(x(t))$ is a martingale as it kills the drift of (22).

- If $\gamma < 1$ ($\alpha > 0$), the state ∞ is a natural inaccessible boundary, by Feller classification of states, [6]. State 0 is an exit (absorbing) state, a regular state or an entrance state depending on $\mu \leq \gamma b^2/2$, $\gamma b^2/2 < \mu < b^2/2$ and $\mu \geq b^2/2$, respectively and also by Feller classification of states. In the first case, the first hitting time of 0 given $x(0) = x > 0$, say $\tau_{x,0}$, is finite a.s.. So extinction occurs with probability 1 in finite time. In the second case, $x(t)$ is self-similar(α), $\alpha = (1 - \gamma)^{-1} > 0$, only if state 0 is made either purely absorbing or purely reflecting. In the last case, if state 0 is considered stationary, then $\tau_{x,0}$, is infinite a.s., [18]: $x(t)$ drifts to ∞ .

Regular and exit boundaries are accessible, while entrance and natural boundaries are inaccessible. The diffusion process reaches a regular boundary with positive probability and it can start afresh from it: one needs to specify the boundary conditions at such a regular boundary point. An exit boundary can also be reached from any starting point in $(0, \infty)$ with positive probability but it is not possible to restart the process from it: the process gets stuck or absorbed at it. The process cannot reach an entrance boundary from any starting point in $(0, \infty)$, but it is possible to restart the process at it. A natural boundary cannot be reached in finite time and it is not allowed to start the process from it.

- If $\gamma = 1$, then (22) is the discriminating process and it coincides with (16). The invariant or speed measure density is $m(x) = b^{-2} x^{2\mu/b^2 - 2}$. The derivative of the scale function is $\phi'(x) = Bx^{-2\mu/b^2} > 0$. By Feller classification of states, the state 0 is always a natural inaccessible boundary. Let $\mu = b^2/2 + c$. State ∞ is an entrance

state or an exit (absorbing) state depending on $\mu < b^2/2$ ($c < 0$) and $\mu > b^2/2$ ($c > 0$), respectively. Thus

$$(24) \quad x(t) \xrightarrow[t \rightarrow \infty]{a.s.} \begin{cases} 0, & \text{if } c < 0 \\ \infty, & \text{if } c > 0 \end{cases}.$$

If $c < 0$, state 0 (extinction) is reached eventually but it cannot be reached in finite time, whereas if $c > 0$, state ∞ is reached in finite time (finite time hyperexponential blowing up). If $c = 0$, both states 0 and ∞ are natural boundaries and the process $x(t)$ is transient: it oscillates indefinitely between the two boundary states. We have $x(t) = e^{s(t)}$ with $s(t)$ obeying: $ds(t) = cdt + bdw(t)$, $s(0) = s$. Here, the discriminating process $x(t) = xe^{(\mu - b^2/2)t + bw(t)}$ is just the geometric Brownian motion with drift c started at $x = e^s$, [27]. Therefore, the probability density starting from $x > 0$ that $x(t)$ is in state $y > 0$ at time $t > 0$ is lognormal with

$$(25) \quad p(x; t, y) = \frac{1}{by\sqrt{2\pi t}} e^{-\frac{1}{2b^2 t}(\log(y/x) - ct)^2}.$$

The most probable state (or the mode) of $x(t)$ given $x(0) = x$ is: $x_*(t, x) = x \exp((\mu - 3b^2/2)t)$, the mean is $\mathbf{E}_x x(t) = x \exp(\mu t)$ and the variance $\text{Var}(x(t)) = x^2 (e^{b^2 t} - 1) \exp(2\mu t) \underset{t \rightarrow \infty}{\sim} x^2 e^{2(\mu + b^2/2)t}$. With $\sigma(x(t)) = \sqrt{\text{Var}(x(t))}$ the standard deviation of $x(t)$, it holds that $\sigma(x(t))/\mathbf{E}x(t) \underset{t \rightarrow \infty}{\sim} e^{b^2 t/2} \xrightarrow[t \rightarrow \infty]{} \infty$, showing that the relative fluctuations of $x(t)$ are exponentially large and that no central limit theorem for $x(t)$ is to be expected. The q -moments of $x(t)$ are

$$(26) \quad \mathbf{E}(x(t)^q) = x^q \exp[(q\mu + q(q-1)b^2/2)t], \quad q > 0,$$

with

$$(27) \quad \mathbf{E}(x(t)^q) \xrightarrow[t \rightarrow \infty]{} \begin{cases} 0, & \text{if } q < 1 - \frac{2\mu}{b^2} \\ \infty, & \text{if } q > 1 - \frac{2\mu}{b^2} \end{cases}.$$

Note that for $-b^2/2 < c < 0$ ($\mu > 0$, $c < 0$): $\mathbf{E}x(t) \xrightarrow[t \rightarrow \infty]{} \infty$ together with $x(t) \xrightarrow[t \rightarrow \infty]{a.s.} 0$. This process $x(t)$ lacks any self-similarity property but it is log-selfsimilar because $x(t, x) = e^{s(t, s)}$ with $s(t, s) = s(t)$ and $s(0) = s$, a selfsimilar process with index $1/2$.

- If $\gamma > 1$ ($\alpha = (1 - \gamma)^{-1} < 0$), the state 0 is a natural inaccessible boundary. State ∞ is an entrance state, a regular state or an exit (or absorbing) state depending on $\mu \leq b^2/2$, $b^2/2 < \mu < \gamma b^2/2$ and $\mu \geq \gamma b^2/2$, respectively. In the first case, the first hitting time $\tau_{x, \infty}$ of ∞ , given $x(0) = x > 0$, is infinite a.s.. In the second case, $x(t)$ is self-similar(α) only if state ∞ is made either purely absorbing or purely reflecting. In the third case, the first hitting time $\tau_{x, \infty}$ of ∞ , given $x(0) = x > 0$, is finite a.s.: explosion occurs with probability 1 in finite time (a case of hyperexponential growth). This results from the following observation: consider the diffusion process (22) with $0 < \gamma < 1$. Consider the change of variables $\bar{x}(t) := x(t)^{-\gamma}$ with state 0 (respectively ∞) mapped to state ∞ (respectively 0). By Itô calculus,

$$(28) \quad d\bar{x}(t) = \bar{\mu}\bar{x}(t)^{\bar{\gamma}} dt + \bar{b}\bar{x}(t)^{(1+\bar{\gamma})/2} dw(t), \quad \bar{x} = x,$$

where $\overline{\gamma} = 1/\gamma > 1$, $\overline{\mu} = \gamma(b^2(\gamma+1)/2 - \mu)$ and $\overline{b} = -b\gamma$. The diffusion process (28) is of the same form as (22) and our conclusions follow the ones obtained in the case $0 < \gamma < 1$ and from the facts: $\mu \geq b^2/2 \Leftrightarrow \overline{\mu} \leq \overline{b}^2/2$ and $\mu \leq \gamma b^2/2 \Leftrightarrow \overline{\mu} \geq \overline{\gamma} \overline{b}^2/2$.

- The critical case ($\mu = 0$): when $\mu = 0$ (or $c = -b^2/2$), $x(t)$ is a martingale so with $\mathbf{E}_x x(t) = x$, constant¹. From the previous study, if $\gamma < 1$, the state ∞ is a natural inaccessible boundary whereas state 0 is exit (or absorbing) and reached eventually in finite time. If $\gamma = 1$ (the discriminating critical process), state ∞ is an entrance state and state 0 a natural inaccessible boundary. Because $c = -b^2/2 < 0$, state 0 (extinction) is reached eventually but now not in finite time. If $\gamma > 1$, state 0 is a natural inaccessible boundary whereas state ∞ is an entrance state. The process drifts to ∞ but not in finite time.

(ii) Consider the following Lamperti spectrally positive process $s(\tau)$:

Take $\pi(dv) = \rho(v)dv$ with $\rho(v) = \kappa v^{-(1+a)}/\Gamma(-a)$, $\kappa > 0$ and $a \in (1, 2)$. Assume $b = 0$ (no Brownian component). Then, when acting on $\phi(s) = e^{-ps}$, $p \geq 0$, the infinitesimal generator of $s(\tau)$ reads

$$(29) \quad G\phi(s) = e^{-ps} \left(\int_0^\infty (e^{-pv} - 1 + pv1_{\{v \leq 1\}}) \pi(dv) - cp \right) = -e^{-ps} \psi(p),$$

where, for some new drift value $\mu = \frac{\kappa a}{\Gamma(2-a)} + c$,

$$(30) \quad \psi(p) = \frac{\kappa}{\Gamma(-a)} \int_0^\infty (1 - e^{-pv} - pv1_{\{v \leq 1\}}) v^{-(1+a)} dv + cp = \mu p - \kappa p^a.$$

The factor $\frac{\kappa a}{\Gamma(2-a)}$ in μ corresponds to an additional drift contribution arising from small jumps in the jump part with density $\rho(v)$. Therefore

$$(31) \quad \psi_\tau(p) := -\log \mathbf{E}_s e^{-p(s(\tau)-s)} = \tau \psi(p).$$

Depending on $c < -\frac{\kappa a}{\Gamma(2-a)}$ or $c > -\frac{\kappa a}{\Gamma(2-a)}$, the global drift is either negative or positive. If $\mu = 0$ (no drift term), we shall speak of the critical Lamperti model. The self-similar process $x(t)$ constructed as a time-changed version of $y(\tau) = e^{s(\tau)}$, with $s(\tau)$ the latter Lamperti process, deserves interest but we shall not run into its detailed study.

(iii) The a -stable subordinator. The jump part of the process $s(\tau)$ can be a subordinator, so with non-decreasing sample paths and with bounded variations, [4]. Taking $\pi(dv) = \rho(v)dv$ with $\rho(v) = \kappa a v^{-(1+a)}/\Gamma(1-a)$, $\kappa > 0$, $a \in (0, 1)$ and $b = 0$, we are led to the one-sided a -stable process with drift. Here therefore, $\psi(p) = \mu p + \kappa p^a$, $\mu = \frac{\kappa a}{\Gamma(2-a)} + c$. The self-similar process $x(t)$ constructed from the latter a -stable process with drift $s(\tau)$ deserves interest but we shall not run into its detailed study either.

¹The use here of the terminology ‘‘criticality’’ refers to whether the process will, on average, decrease $\mu < 0$ (subcriticality), remain constant $\mu = 0$ (criticality) or increase $\mu > 0$ (supercriticality).

4. GROWTH PROCESSES AS CONTINUOUS-STATE BRANCHING PROCESSES (CSBPs)

4.1. Generalities on CSBPs. Let $s(\tau)$ be the above spectrally positive Lévy process defined by (11). Following the time-change suggested in (ii) of subsection 2.2, consider now the new time substitution: $t_\tau = \int_0^\tau s(\tau')^{-1} d\tau'$, defined up to the first hitting time of 0 of $s(\tau)$. Then its inverse is $\tau_t = \int_0^t x(s) ds$ where $x(t) := s(\tau_t) = s\left(\int_0^t x(s) ds\right)$. Therefore, $x(t)$ with $x(0) = x$, solves the stochastic differential equation (SDE)

$$(32) \quad x(t) = x + c \int_0^t x(s) ds + b \int_0^t \sqrt{x(s)} dw(s) + \int_0^t \int_0^\infty \int_0^{x(t-)} v \tilde{N}(ds, dv, dx),$$

where $w = (w(t), \tau \geq 0)$ is a standard Brownian motion, $N(ds, dv, dx)$ is a Poisson random measure with intensity $ds \cdot \pi(dv) \cdot dx$ independent of w and \tilde{N} is the compensated measure of N . And $x(t)$ is a continuous-state branching process (CSBP), [17], stopped when it first hits 0 if ever. From [7] indeed, a CSBP can also be defined as the unique non-negative strong solution of this SDE. CSBPs may be viewed as properly scaled versions of the classical integral-valued branching processes, [19], [10], [4].

Suppose $x(0) = x = 1$. Let then $\Psi_t(p) := -\log \mathbf{E}_{x=1} e^{-px(t)}$, the log-Laplace transform (LLt) of $x(t)$. Then [17], $\Psi_t(p)$ obeys

$$(33) \quad \dot{\Psi}_t(p) = \psi(\Psi_t(p)), \quad \Psi_0(p) = p,$$

with ψ given by (11) known as the branching mechanism of $x(t)$. We clearly have

$$(34) \quad \Psi_t(p) = B^{-1}(t + B(p)), \quad \text{where } B(p) = \int^p \frac{dq}{\psi(q)}.$$

Furthermore, with

$$(35) \quad \Psi_{t,x}(p) := -\log \mathbf{E}_x e^{-px(t)}, \quad \Psi_{t,x}(p) = x\Psi_t(p).$$

Depending on $\psi'(0^+)$ positive, zero or negative, $x(t)$ is supercritical, critical or subcritical. In the supercritical case, $x(t)$ started at $x > 0$ has a positive extinction probability $\rho_{x,\text{ext}} = \rho_{\text{ext}}^x$ with $\rho_{\text{ext}} := \rho_{1,\text{ext}} = \exp(-p_c)$ and p_c the largest solution to $\psi(p) = 0$. If in the supercritical case $\psi(p) \geq 0$ for all $p \geq 0$, by convention $p_c = \infty$ and therefore $\rho_{x,\text{ext}} = 0$ (a case of strict supercriticality). In the critical and subcritical cases, $x(t)$ started at $x > 0$ goes extinct with probability 1.

If $\tau_{x,0} = \inf(t > 0 : x(t) = 0 \mid x(0) = x)$ now denotes the time to extinction, we have

$$(36) \quad \mathbf{P}(\tau_{x,0} \leq t) = e^{-x\Psi_t(\infty)}.$$

4.2. Examples. We shall consider 3 fundamental examples:

- $\rho \equiv 0$. We are led to the Feller diffusion on $[0, \infty)$ (compare with (16)):

$$(37) \quad dx(t) = cx(t) dt + b\sqrt{x(t)} dw(t), \quad x(0) = x = 1.$$

We shall let $f(x) = cx$, the drift and $g(x) = bx^{1/2}$, the local volatility. Note $g(x)$ is non-Lipschitz, so singular, as x approaches 0. The invariant or speed measure density of this diffusion process is $m(x) = g^{-2}(x) \exp 2 \int^x f/g^2(y) dy = b^{-2}x^{-1}e^{2cx/b^2}$. Its scale or harmonic function is $\phi(x) = A + B \int^x dy \exp(-2 \int^y f/g^2(z) dz)$, for some constants $A, B > 0$, so with $\phi'(x) = B \exp(-2 \int^x f/g^2(y) dy) = Be^{-2cx/b^2} > 0$. It is such that $\phi(x(t))$ is a martingale. By Feller classification of states, whatever the values of c , state 0 is absorbing, whereas state ∞ is an inaccessible natural boundary, [5].

Here $\psi(p) = cp - \frac{1}{2}b^2p^2$ and with $\Psi_t(p) := -\log \mathbf{E}_1 e^{-px(t)}$, then $\Psi_t(p)$ obeys $\dot{\Psi}_t(p) = \psi(\Psi_t(p))$, $\Psi_0(p) = p$, $\Psi_{t,x}(p) = x\Psi_t(p)$. This can be solved to give

$$(38) \quad \Psi_t(p) = \begin{cases} pe^{ct} / (1 + (b^2p/(2c))(e^{ct} - 1)) & \text{if } c \neq 0 \\ (2p) / (2 + b^2tp) & \text{if } c = 0 \end{cases}.$$

We note that, when $c = 0$, $\Psi_{\lambda t}(\lambda^{-1}p) = \lambda^{-1}\Psi_t(p)$, a self-similarity property. Thus, $\Psi_{\lambda t, \lambda x}(\lambda^{-1}p) = \Psi_{t,x}(p)$ and, with $x(t, x)$ the solution of (37) with initial condition $x(0) = x$, $x(\lambda t, \lambda x) \stackrel{d}{=} \lambda x(t, x)$ ²: the critical Feller diffusion is self-similar with index $\alpha = 1$.

The case $c < 0$ ($c > 0$) corresponds to a subcritical (supercritical) Feller CSBP. $c = 0$ is the critical case with $x(t)$ being a martingale. We have:

$$(39) \quad \mathbf{E}_x(x(t)) = x\Psi'_t(0) = \begin{cases} xe^{ct} & \text{if } c \neq 0 \\ x & \text{if } c = 0 \end{cases}.$$

- In the supercritical case with $c > 0$, the extinction probability of $x(t)$ given $x(0) = x$ is $\rho_{x,\text{ext}} = \exp(-xp_c) = \exp(-2xc/b^2)$ and the law of the time to extinction $\tau_{x,0}$ given $x(0) = x$ is

$$(40) \quad \mathbf{P}(\tau_{x,0} \leq t) = \exp -x [(b^2/(2c))(1 - e^{-ct})]^{-1},$$

with exponential tails: $e^{ct}\mathbf{P}(\tau_{x,0} > t) \rightarrow \text{constant}$ as $t \rightarrow \infty$. If $c > 0$, the law of $\tau_{x,0}$ has an atom at $t = \infty$ with mass $1 - \exp(-2xc/b^2)$, corresponding to the probability that $x(t)$ drifts to ∞ . If the latter event occurs, it cannot be in finite time.

- If $c \leq 0$ (sub- and critical case), $\rho_{x,\text{ext}} = 1$ and $x(t)$ hits 0 with probability 1 and stays there for ever. The law of the time to extinction $\tau_{x,0}$ given $x(0) = x$ in this case is

$$(41) \quad \mathbf{P}(\tau_{x,0} \leq t) = \begin{cases} \exp -x [(b^2/(-2c))(e^{-ct} - 1)]^{-1} & \text{if } c < 0 \\ \exp -2x/(b^2t) & \text{if } c = 0 \end{cases}.$$

- In the subcritical case ($c < 0$), tails are exponential: $e^{-ct}\mathbf{P}(\tau_{x,0} > t) \rightarrow \text{constant}$ as $t \rightarrow \infty$. In the critical case ($c = 0$), the law of $\tau_{x,0}$ is tail-equivalent to $2x/(b^2t)$ in that $\frac{b^2t}{2x}\mathbf{P}(\tau_{x,0} > t) \rightarrow 1$ as $t \rightarrow \infty$; thus, $\tau_{x,0}$ has Pareto-like heavy tails and the time to extinction is thus much longer statistically than when $c < 0$.

• $b = 0$ and $\pi(dv) = \rho(v)dv$ with $\rho(v) = \kappa v^{-(1+a)}/\Gamma(-a)$, $\kappa > 0$ and $a \in (1, 2)$. We are then led to the Lamperti CSBP process $x(t)$, [17].

²This property can easily be extended to all finite-dimensional distributions.

Let $\Psi_t(p) := -\log \mathbf{E} e^{-p\mathbf{x}(t)}$. Then $\Psi_t(p)$ obeys $\dot{\Psi}_t(p) = \psi(\Psi_t(p))$, $\Psi_0(p) = p$ where $\psi(p) = \mu p - \kappa p^a$, $\mu = \frac{\kappa a}{\Gamma(2-a)} + c$. It is a CSBP, with here

$$(42) \quad \Psi_t(p) = \begin{cases} (p^{1-a} e^{-\mu(a-1)t} + (\kappa/\mu) (1 - e^{-\mu(a-1)t}))^{-1/(a-1)} & \text{if } \mu \neq 0 \\ (p^{1-a} + \kappa(a-1)t)^{-1/(a-1)} & \text{if } \mu = 0 \end{cases},$$

and $\Psi_{t,x}(p) = x\Psi_t(p)$. Classical (i.e. discrete-space, continuous-time Bienaymé-Galton-Watson) branching processes displaying similar properties with finite mean and infinite variance were considered in [28] and [2].

The case $\mu < 0$ ($\mu > 0$) corresponds to a subcritical (supercritical) Lamperti CSBP. $\mu = 0$ is the critical case with $x(t)$ a martingale.

Note that $\Psi_t(p) \xrightarrow[p \rightarrow 0^+]{p \rightarrow 0^+} 0$ for all $t > 0$. The Lamperti CSBP is regular or conservative with $\mathbf{P}(x(t) < \infty) = 1$.

In the critical case when $\mu = 0$, with $\alpha := 1/(a-1) > 1$, $\Psi_{\lambda t}(\lambda^{-\alpha} p) = \lambda^{-\alpha} \Psi_t(p)$, a self-similarity property. And indeed, $\Psi_{\lambda t, \lambda^\alpha x}(\lambda^{-\alpha} p) = \Psi_{t,x}(p)$, showing that $x(\lambda t, \lambda^\alpha x) \stackrel{d}{=} \lambda^\alpha x(t, x)$, a self-similarity property with index $\alpha > 1$ for $x(t)$.

Here, $x(t)$ is the jump process with drift

$$(43) \quad dx(t) = cx(t) dt + \kappa x(t_-)^{1/a} ds(t), \quad x(0) = x = 1,$$

where $s(t)$ is the driving a -stable spectrally positive Lévy process ($a \in (1, 2)$), with no superposed driving Brownian component. For this model, $\rho_{x,\text{ext}} = \exp(-xp_c) = \exp(-x(\mu/\kappa)^{1/(a-1)})$ in the supercritical case $\mu > 0$ (1 otherwise) and

$$(44) \quad \mathbf{P}(\tau_{x,0} \leq t) = e^{-x\Psi_t(\infty)} = \begin{cases} \exp -x \left(\frac{\kappa}{\mu} (1 - e^{-\mu(a-1)t}) \right)^{-\frac{1}{a-1}}, & \mu \neq 0 \\ \exp -x (\kappa(a-1)t)^{-\frac{1}{a-1}}, & \mu = 0 \end{cases}.$$

If $c > 0$, the law of $\tau_{x,0}$ has an atom at $t = \infty$ with mass $1 - \exp(-x(\mu/\kappa)^{1/(a-1)})$, the probability of explosion $\rho_{x,\text{exp}} = 1 - \rho_{x,\text{ext}}$.

In the critical case ($\mu = 0$), the law of $\tau_{x,0}$ is tail-equivalent to $x(\kappa(a-1)t)^{-1/(a-1)}$ as $t \rightarrow \infty$. Thus, $\tau_{x,0}$ has power-law heavy tails and the time to extinction is thus longer statistically than when $\mu < 0$.

We can condition the critical model on non-extinction and compute $\mathbf{E}_1(x(t) \mid x(t) > 0)$. Indeed, we have ([26], Theorem 1), conditionally given $x(t) > 0$,

$$(45) \quad \mathbf{P}_1(\tau_{1,0} > t) \cdot x(t) \xrightarrow[t \rightarrow \infty]{d} W,$$

where the random variable W has $\text{LSt } \mathbf{E}(e^{-pW}) = 1 - (1 + p^{-(a-1)})^{-1/(a-1)}$, therefore with finite mean 1. $x(t)$ has a quasi-stationary regime, [32]. We have $\mathbf{P}(\tau_{1,0} \leq t) = e^{-\Psi_t(\infty)}$ and so $\mathbf{P}(\tau_{1,0} > t) = 1 - e^{-\Psi_t(\infty)} \sim \Psi_t(\infty)$. This shows that as t gets large

$$(46) \quad \mathbf{E}_1(x(t) \mid x(t) > 0) \sim \frac{-1}{\Psi_t(\infty)} \partial_p (\Psi_t(\infty) - \Psi_t(p)) \big|_{p=0} = \frac{1}{\Psi_t(\infty)} = (\kappa(a-1)t)^{1/(a-1)},$$

displaying slow algebraic superlinear growth in time, with exponent $\alpha = 1/(a-1) > 1$.

• Taking $\pi(dv) = \rho(v)dv$ with $\rho(v) = \kappa av^{-(1+a)}/\Gamma(1-a)$, $\kappa > 0$, $a \in (0,1)$, $b = 0$, we are led to the standard one-sided a -stable subordinator process $s(\cdot)$ with drift. Note that π now integrates $1 \wedge v$: small jumps are less likely than in the a -stable spectrally positive case with $a \in (1,2)$, but large jumps of the one-sided a -stable subordinator are more likely to occur than in the spectrally positive case. For this model, $\psi(p) = \mu p + \kappa p^a$, $\mu = \frac{\kappa a}{\Gamma(2-a)} + c$. The $\Psi_t(p)$ solving (33) of the corresponding CSBP is seen to be

$$(47) \quad \Psi_t(p) = \begin{cases} (p^{1-a} e^{\mu(1-a)t} + (\kappa/\mu)(e^{\mu(1-a)t} - 1))^{1/(1-a)} & \text{if } \mu \neq 0 \\ (p^{1-a} + \kappa(1-a)t)^{1/(1-a)} & \text{if } \mu = 0 \end{cases},$$

and $\Psi_{t,x}(p) = x\Psi_t(p)$. We note that, when $\mu = 0$, $\Psi_{\lambda t}(\lambda^{1/(1-a)}p) = \lambda^{1/(1-a)}\Psi_t(p)$, a self-similarity property. With $\alpha = 1/(a-1)$, we have $\Psi_{\lambda t, \lambda^\alpha x}(\lambda^{-\alpha}p) = \Psi_{t,x}(p)$ showing that $x(\lambda t, \lambda^\alpha x) \stackrel{d}{=} \lambda^\alpha x(t, x)$, a self-similarity property with index $\alpha < -1$. We have,

$$(48) \quad \Psi_t(p) \xrightarrow{p \rightarrow 0^+} \begin{cases} ((\kappa/\mu)(e^{\mu(1-a)t} - 1))^{1/(1-a)} & \text{if } \mu \neq 0 \\ (\kappa(1-a)t)^{1/(1-a)} & \text{if } \mu = 0 \end{cases},$$

and, the limit being non zero for all $t > 0$, this CSBP is non-conservative as it loses mass at ∞ instantaneously, with $\mathbf{P}_x(x(t) < \infty) = e^{-x\Psi_t(0)}$. This is a consequence of $\int_{0^+} dq/\psi(q) < \infty$ leading to this superexponential growth situation. Here, $x(t)$ is the jump process with drift

$$(49) \quad dx(t) = cx(t)dt + \kappa x(t_-)^{1/a} ds(t), \quad x(0) = x = 1,$$

where $s(t)$ is the driving a -stable subordinator ($a \in (0,1)$) with no Brownian component. For this supercritical model with $\psi'(0^+) = \infty$, $\rho_{x,\text{ext}} = \exp(-xp_c) = \exp(-x(-\mu/\kappa)^{1/(a-1)})$ if $\mu < 0$ (0 otherwise) and $\tau_{x,0} = \infty$ with probability 1 as a result of $B(p) = \int^p dq/\psi(q) \xrightarrow{p \rightarrow \infty} \infty$ leading to $\Psi_t(p) \xrightarrow{p \rightarrow \infty} \infty$. If $\mu \geq 0$ indeed, $\psi(p)$ stays positive with $\psi(p) \xrightarrow{p \rightarrow \infty} \infty$ with by convention $p_c = \infty$ and so $\rho_{x,\text{ext}} = 0$.

• The critical growth case: the Neveu model. It remains to consider the case $a \rightarrow 1$.

- Considering the branching mechanism of the 1-sided a -stable subordinator: $\psi(p) = \mu p + \kappa p^a$, $a \in (0,1)$, $\kappa > 0$ (respectively the one of the a -stable Lamperti spectrally positive Lévy process: $\psi(p) = \mu p - \kappa p^a$, $a \in (1,2)$, $\kappa > 0$) and letting simply $a \rightarrow 1^-$ (respectively $a \rightarrow 1^+$), we are led to the branching mechanism of the pure drift model $\psi(p) = (\mu + \kappa)p$ (respectively $\psi(p) = (\mu - \kappa)p$). The corresponding CSBP is Malthusian and trivial: $x(t, x) = xe^{(\mu+\kappa)t}$ (respectively $x(t, x) = xe^{(\mu-\kappa)t}$). This process lacks any self-similarity property.

There is a more interesting way to take the limits $a \rightarrow 1^\mp$:

- Consider the branching mechanism of the 1-sided a -stable subordinator: $\psi(p) = \mu p + \kappa p^a$, $a \in (0,1)$, $\kappa > 0$. Define the constants μ' , $\kappa' > 0$ by $\mu = \mu' - \kappa$ and

$\kappa = \kappa' / (1 - a)$. Then, as $a \rightarrow 1^-$, $\mu \rightarrow -\infty$ and $\kappa \rightarrow +\infty$ in a suitable way. And ψ reads $\psi(p) = \mu'p - \frac{\kappa'}{1-a}p(1 - p^{a-1}) \sim \mu'p - \kappa'p \log p$.

- Consider the branching mechanism of the a -stable Lamperti spectrally positive Lévy process: $\psi(p) = \mu p - \kappa p^a$, $a \in (1, 2)$, $\kappa > 0$. Define μ' , $\kappa' > 0$ by $\mu = \mu' + \kappa$ and $\kappa = \kappa' / (a - 1)$. Then, as $a \rightarrow 1^+$, both μ , κ tend to $+\infty$. And ψ reads $\psi(p) = \mu'p + \frac{\kappa'}{a-1}p(1 - p^{a-1}) \sim \mu'p - \kappa'p \log p$.

The CSBP with new branching mechanism, say $\psi(p) = \mu p - \kappa p \log p$, $\kappa > 0$, is the Neveu CSBP, [23]. $\mu < 0$, $\mu = 0$ and $\mu > 0$ correspond respectively to the subcritical, critical and supercritical versions of the Neveu process. Note $\psi'(0^+) = +\infty$, so that $\mathbf{E}_x(x(t)) = +\infty$ and it may be shown, using martingale arguments [23], that, conditionally given $x(t)$ drifts to ∞ , it does so at double exponential speed a.s.. So if the population does not go extinct, $x(t)$ grows fast to infinity at a double-exponential speed: $e^{-\kappa t} \log x(t) \xrightarrow{d} E > 0$ as $t \rightarrow \infty$, with E standard exponentially distributed: $\mathbf{P}(E > x) = e^{-x}$, $x > 0$. Using martingale arguments, this convergence can be shown to be almost sure as well, ([9], [23], [11], Proposition 3.8).

The LLt $\Psi_t(p)$ of the corresponding CSBP solving (33) is easily seen to be

$$(50) \quad \Psi_t(p) = \begin{cases} \exp\left(\frac{\mu}{\kappa}(1 - e^{-\kappa t})\right) p^{e^{-\kappa t}} & \text{if } \mu \neq 0 \\ p^{e^{-\kappa t}} & \text{if } \mu = 0 \end{cases}.$$

The marginal distribution of the critical Neveu CSBP is one-sided $e^{-\kappa t}$ -stable. It holds that $\Psi_t(p) \xrightarrow{p \rightarrow 0^+} 0$ for all $t > 0$ and the critical Neveu CSBP is regular or conservative, with $\mathbf{P}_1(x(t) < \infty) = 1$. It can be shown that, for the critical Neveu process, $x(t) \xrightarrow[t \rightarrow \infty]{a.s.} 0$ (extinction a.s.), but not in finite time, [12]. We observe that the critical version of the Neveu model lacks any self-similarity space/time property.

4.3. Summary. Let us summarize our results:

We considered mainly 3 fundamental CSBPs $x(t)$: the Feller diffusion model ($a = 2$), the a -Lamperti CSBP ($a \in (1, 2)$) and the one-sided a -stable CSBP ($a \in (0, 1)$):

The critical version of these models were shown to exhibit self-similarity properties: the obtained Hurst indices are $\alpha = 1$, $\alpha = 1/(a - 1) > 1$ and $\alpha = 1/(a - 1) < -1$, respectively. To some extent, the Feller diffusion model may be viewed as the limiting situation $a \rightarrow 2^-$ of the Lamperti CSBP.

Taking $a \rightarrow 1^\mp$ yields in the first place the deterministic Malthusian growth models: $x(t) = xe^{(\mu \mp \kappa)t}$. This Malthusian regime separates a situation for which conditionally given $x(t) > 0$, the mean of $x(t)$ has superlinear algebraic growth rate (for the a -Lamperti model, see (46)) and a situation for which $x(t)$ is not regular as it blows up for all time $t > 0$ (for the one-sided a -stable model). It is the discriminating critical process of such CSBP population growth models. This should be compared with similar behaviors obtained in the deterministic setup. A main difference of the stochastic dynamics as compared to the deterministic case is that all critical CSBPs go extinct with probability 1.

While considering a different limiting process as $a \rightarrow 1^\mp$, we obtained the Neveu CSBP model which grows a.s. at double superexponential speed. The critical version of this process is no longer self-similar. It plays the role of the superexponential discriminating deterministic model separating two log-self-similar models: the exp-algebraic rate model and the blowing-up model, respectively.

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REFERENCES

- [1] Andersen, J.V.; Sornette, D. Fearless versus Fearful Speculative Financial Bubbles. *Physica A* 337 (3-4), 565-585, (2004).
- [2] Avan, J.; Grosjean, N.; Huillet, T. On extreme events for non-spatial and spatial branching Brownian motions. *Physica D: Nonlinear Phenomena*, Volume 298, 13-20, (2015).
- [3] Avila, Piret, magistrikraad, (juh) Rekker, Astrid. Stochastic Super-Exponential Growth Model in Population Dynamics, Phd thesis, Tallin University, 2011. <https://www.stat.ee/dokumendid/57943>
- [4] Bertoin, J. *Subordinators, Lévy processes with no negative jumps and branching processes*. Lecture Notes of the Concentrated Advanced Course on Lévy Processes, Maphysto, Centre for Mathematical Physics and Stochastics, Department of Mathematical Sciences, University of Aarhus, 2000.
- [5] Feller, W. Two singular diffusion problems. *Ann. of Math.* 54, 173-182, (1951).
- [6] Feller, W. The parabolic differential equations and the associated semi-groups of transformations. *Ann. of Math.* (2) 55, 468-519, (1952).
- [7] Fu, Z.; Li, Z. Stochastic equations of non-negative processes with jumps. *Stochastic Processes and their Applications*. Volume 120, Issue 3, Pages 306-330, (2010).
- [8] Goriely, A.; Hyde, C. Necessary and sufficient conditions for finite-time singularities in ordinary differential equations. *Journal of Differential Equations*, 161, 422-448, (2000).
- [9] Grey, D. R. Almost sure convergence in Markov branching processes with infinite mean. *J. Appl. Probability*, 14(4), 702-716, (1977).
- [10] Harris, T. E. *The theory of branching processes*. Die Grundlehren der Mathematischen Wissenschaften, Bd. 119 Springer-Verlag, Berlin; Prentice-Hall, Inc., Englewood Cliffs, N.J. 1963.
- [11] Hénard, O. The fixation line in the Lambda-coalescent. *Ann. Appl. Prob.*, Volume 25, Number 5, 3007-3032, (2015).
- [12] Huillet, T. Energy cascades as branching processes with emphasis on Neveu’s approach to Derrida’s random energy model. *Adv. in Appl. Probab.* 35(2), 477-503, (2003).
- [13] Hüsler, A. D.; Sornette, D. Human population and atmospheric carbon dioxide growth dynamics: Diagnostics for the future. *The European Physical Journal: Special Topics*. Volume 223, Issue 11, 2065-2085, (2014).
- [14] Johansen, A.; Sornette, D. Finite-time singularity in the dynamics of the world population, economic and financial indices. *Physica A* 294 (3-4), 465-502, (2001).
- [15] Kolmogorov, A. N. Wiener’sche Spiralen und einige andere interessante Kurven im Hilbertschen Raum. *Doklady A.N., S.S.S.R.* (n.s.) 26, 115-118, (1940).
- [16] Kyprianou, A. E. *Introductory Lectures on Fluctuations of Lévy Processes with Applications*. Universitext Springer, Second Edition, 2014.
- [17] Lamperti, J. W. Continuous state branching processes. *Bull. of the Am. Math. Soc.*, 73, 382-386, (1967).
- [18] Lamperti, J. W. Semi-stable stochastic processes. *Z. Wahrscheinlichkeitstheorie verw. Geb.* 22, 205-225, (1972).

- [19] Li, Z. Continuous state branching processes. arXiv:1202.3223, (2012).
- [20] Li Lin; Sornette, D. Diagnostics of rational expectation financial bubbles with stochastic mean-reverting termination times. *The European Journal of Finance* 19 (5-6) 344-365 (2013).
- [21] Li Lin; Ren R.E; Sornette, D. The volatility-confined LPPL Model: A consistent model of ‘explosive’ financial bubbles with mean-reversing residuals. *International Review of Financial Analysis* 33, 210-225 (2014).
- [22] Mandelbrot, B.; Van Ness, J. W. Fractional Brownian motions, fractional noises and applications. *SIAM. Review* 10, 422-437, (1968).
- [23] Neveu, J. A continuous state branching process in relation with the GREM model of spin glass theory. Unpublished Technical Report 267, Ecole Polytechnique, (1992).
- [24] Reimann, St.; Gontis, V.; Alaburda, M. Interplay between positive feedbacks in the generalized CEV process. *Physica A* 390, 1393-1401, (2001).
- [25] Romer, P. M. The Origins of Endogenous Growth. *The Journal of Economic Perspectives*, Vol. 8, No. 1, 3-22, (1994).
- [26] Ren, Y-X.; Yang, T.; Zhao, G-H. Conditional limit theorems for critical continuous-state branching processes. *Science China, Mathematics*, Vol. 57 No. 12, 2577-2588, (2014).
- [27] Ross, S. M. *Variations on Brownian Motion*. Introduction to Probability Models (11th ed.). Amsterdam: Elsevier. pp. 612-614, 2014.
- [28] Saichev, A.; Sornette, D. Super-linear scaling of offsprings at criticality in branching processes. *Phys. Rev. E* 89, 012104, (2014).
- [29] Sornette, D.; Andersen, J. V. A Nonlinear Super-Exponential Rational Model of Speculative Financial Bubbles. *Int. J. Mod. Phys. C*, 13, 2, 171-187, (2002).
- [30] Sornette, D.; Helmstetter, A. Occurrence of Finite-Time-Singularity in Epidemic Models of Rupture, Earthquakes and Starquakes. *Physical Review Letters* 89 (15), 158501, (2002).
- [31] Varfolomeyev S. D.; Gurevich K. G. The hyperexponential growth of the human population on a macrohistorical scale. *J. Theor. Biol.* 7; 212(3), 367-372, (2001).
- [32] Yaglom, A. M. Certain limit theorems of the theory of branching stochastic processes. *Doklady Akademii Nauk SSSR*, 56, 795-798, (1947).

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